2 Doing calculations and representing values

Key words: unit, quantity, compound measure, base unit, derived unit, variable, decimal, fraction, significant figures, round, integer, recurring decimal, decimal place, mean, arithmetic mean, index notation, index, power, exponent, square, cube, square root, cube root, reciprocal, unit prefix, standard form, standard index form, scientific notation, power of 10, order of magnitude, approximation, estimate.

The value of a quantity such as temperature or mass is represented by a *number* and a *unit*. This chapter focuses on the ways that units are used in calculations and on how values are represented. When pupils are using values in calculations, it is important that they are also thinking about the meaning of what they are calculating (see Section 9.5 *The real-world meaning of a formula* on page 93).

2.1 Calculations and units

Doing calculations on values involves paying attention to the manipulation of not just the numbers but the ***units*** as well. Addition and subtraction of values can only be done if they are expressed in the *same units*. For example, it may make sense to add the masses of two objects together (say 15 g and 20 g) to give a total mass (35 g). It would not make sense to add the mass (in g) of one object to the length (in m) of another. Mass and length are different kinds of ***quantity*** and so cannot be added together. However, it would be possible to add the values of a mass (in g) to another mass (in ounces) if they are converted to a common unit, since they are the same kind of quantity.

If some water at 60 °C is added to some water at 20 °C, it does not make sense to add the temperatures together, even though they are the same kind of quantity expressed in the same units. The masses of the water can be added together because mass is an *extensive property* (dependent on the size of the system), but temperature is an *intensive property* (independent of the size of the system) and cannot be added in this way. It would, however, make sense to calculate the temperature rise of an object (in °C) by subtracting an initial temperature (in °C) from a final temperature (in °C).

Multiplication and division may involve *different units*. For example, if a ball rolls 8 metres along the ground in 2 seconds, its average speed can be calculated. Here, the division has been done in two steps for emphasis – first the units and then the numbers.

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{8\ \text{m}}{2\ \text{s}} = \frac{8}{2}\ \text{m/s} = 4\ \text{m/s}$$

In mathematics, a quantity such as speed (e.g. in metres per second) is called a ***compound measure*** – it involves two measures of different types; in this case, distance and time. In science, the unit 'metres per second' or 'm/s' would be called a ***derived unit***. In the International System of Units (SI), there are seven ***base units*** (metre, kilogram, second, ampere, kelvin, mole, candela) from which all other units are derived. Some derived units are expressed in terms of the base units (such as m/s); other derived units are given special names (e.g. the unit of force is derived from the base units but is given the name 'newton').

In science, it is good practice always to include units as part of the calculation, in order to keep track of what the numbers mean. An example of a multiplication involving a derived unit would be to calculate the mass of 10 cm^3 of ethanol (density 0.79 g/cm^3).

$$\begin{aligned}\text{mass} &= \text{volume} \times \text{density}\\ &= 10\ \text{cm}^3 \times 0.79\ \text{g/cm}^3\\ &= 7.9\ \text{g}\end{aligned}$$

Multiplying 10 by 0.79 gives 7.9, and multiplying cm^3 by g/cm^3 gives g (grams). Since this is an appropriate unit for mass, it provides a check that the calculation has been done correctly. It also acts as a check that a formula has been written down or rearranged correctly.

Note that not all quantities have units. Those that are derived from a ratio of the sizes of two quantities do not have units, for example relative atomic mass or refractive index.

Calculations involving chemical amounts (in moles) can often lead to confusion over the use of units. For example: What is the mass of 2 mol of water molecules? The relative molecular mass of water is 18, but it is not correct to say that the mass is $2 \times 18 = 36$ g, since the units are not consistent (the relative molecular mass has no units). It is the molar mass of water (18 g/mol) that is needed for the calculation.

$$\begin{aligned}\text{mass} &= \text{chemical amount} \times \text{molar mass}\\ &= 2\ \text{mol} \times 18\ \text{g/mol}\\ &= 36\ \text{g}\end{aligned}$$

In post-16 physics, this kind of checking of consistency of units becomes even more important, and is known as *dimensional analysis*.

Note that, in mathematics, units in calculations are handled differently. In the above formulae, the ***variables*** (mass, volume, and so on) represent *values with units*, so these are part of the calculations. In mathematics, however, the variables in equations do not have units. For example, if the mass of an object is being calculated from an algebraic formula, one might represent the mass as m kg. Here, the variable 'm' represents *just a number*. If the result of the calculation is $m = 6$ then the mass of the object is 6 kg. Teachers and pupils need to be aware of this difference in the way that units are handled in mathematics and science.

2.2 Fractions and decimals

In scientific calculations, intermediate and final values are usually expressed as ***decimals*** rather than ***fractions***. In mathematics, pupils learn to add, subtract, multiply and divide fractions, though this is not much used in science.

One reason for this is that, when dealing with ***integers*** in mathematics, it makes sense to be able to manipulate the number to produce a result expressed as a fraction that also involves integers (below left). This fraction is exact, whereas expressing this value as a decimal would, in this case, not be exact. Using numbers in this way also helps to develop an understanding of how algebraic expressions can be manipulated (below right).

$$\frac{5}{3}+\frac{3}{4}=\frac{20+9}{12}=\frac{29}{12} \qquad \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

This kind of algebraic manipulation of fractions would not be used in secondary science, however, when doing calculations with *measured values*. The values obtained from measurements are not exact integers, so the emphasis is on convenience of calculation. For a multi-step problem, it is generally easier to calculate intermediate values at each step, rather than to build up an expression, leaving the calculation to the last step.

2.3 Rounding and significant figures

The values used in a calculation may not all have the same number of ***significant figures***. For a measured value, the number of significant figures is an indication of the *precision* of measurement. For a calculated value, the number of significant figures should reflect the precision of the values used in the calculation (see Section 1.2 *Measurement, resolution and significant figures* on page 9).

A rule of thumb for rounding

A useful rule of thumb is to ***round*** the result to the same number of significant figures as the measured value with the fewest significant figures. This means that the precision of the result is determined by the least precise value used in the calculation.

For example, to calculate the distance travelled in 2.73 seconds by a ball with velocity 1.4 m/s:

$$\text{distance} = \text{velocity} \times \text{time} = 1.4\,\text{m/s} \times 2.73\,\text{s} = 3.822\,\text{m} = 3.8\,\text{m}$$

The number obtained by multiplying 1.4×2.73 is 3.822, but this is then rounded to two significant figures (3.8). This is because the number in the calculation with the fewer significant figures is 1.4 (two significant figures). Rounding means replacing the calculated value with the nearest number with the appropriate number of significant figures; if the calculated value is halfway between two values with the appropriate number of significant figures then it is rounded up (e.g. 3.85 rounded to two significant figures is 3.9). Older pupils may also be introduced to the convention of writing '(to 2 s.f.)' after the final value in the above calculation. This makes explicit how the result was rounded, and also avoids the implication that two unequal values are equal (i.e. 3.822 m = 3.8 m).

Distinguishing between measured values and integers

Integers need to be handled in a different way. For example, the height of an A4 sheet of paper is 297 mm. The height of 2 sheets placed end-to-end is 594 mm (2×297 mm). This is *not* rounded to 600 mm, since the value '2' is not treated as having only one significant figure. It is an *integer*, and it is *exactly* 2 (in a sense, it has an infinite number of significant figures: 2.000 000 000 . . .). The 'number of sheets' is a 'count' and not a 'measurement'.

Recurring decimals

Sometimes, in calculations involving division, the numerator divides exactly by the denominator (e.g. 18 / 2.4 = 7.5). If not, the result will be a ***recurring decimal*** (e.g. 26 / 2.4 = 10.833 333 33 . . .), even if the recurring pattern is not apparent because the calculator display does not have sufficient digits to show this (e.g. 26/2.3   11.304 347 83…). In mathematics, recurring decimals can be represented by placing dots over the digits; this convention is not needed in science, since such results are rounded to an appropriate number of significant figures.

The meaning of zeros in a value

It is important to pay attention to the way the zero digit is used to indicate the number of significant figures for measured values and for the results from calculations. For example, if a ball with a velocity 2.0 m/s travels for 4.32 seconds, the distance is found by multiplying these two values together. The calculated value is 8.64 m. The zero digit in '2.0 m/s' means that this value has two significant figures. Rounding the calculated value to two significant figures gives 8.6 m.

Similarly, if a ball with a velocity 1.4 m/s travels for 2.86 seconds, multiplying these values together gives a result for the distance of 4.004 m. Rounding this to two significant figures gives 4.0 m. Writing this as 4 m would mean something different – it would have only one significant figure and would indicate less precision in the result.

The use of the zero digit in numbers that do not have a decimal point can be ambiguous. For example, while stating a distance as 5837 m implies that it has been measured or calculated to the nearest metre, it is not so clear what 6300 m means. Does it mean that it has only been measured to the nearest 100 m? (This would imply that the true value is nearer to 6300 m than to 6200 m or 6400 m.) Or to the nearest 10 m? Or to the nearest 1 m? Without knowing the context, it is difficult to interpret what these values mean. One solution is to re-express the value in a different unit. For example, in this case, if km were used, the difference between 6.300 km and 6.3 km would be clear. Another solution is to express the value using standard form (see Section 2.6 *Dealing with very large and very small values* on page 20).

Using judgement when adding values

Judgement is necessary in using the rule of thumb when adding values, as the two examples below illustrate.

If the mass of a coin is 7.17 g then the mass of two such coins would be best expressed as 14.34 g (and not rounded to 14.3 g, even though the original value only had three significant figures). In this case, it makes sense to keep the number of ***decimal places*** the same, since this reflects the resolution of the measuring instrument.

Another example where it makes sense to consider decimal places rather than the number of significant figures would be in finding the total mass of two objects with masses of 1.24 g and 141.5 g. These values suggest that the first object was measured using a higher resolution instrument (to the nearest 0.01 g) than the second object (to the nearest 0.1 g). When the two values are added together the result should be given to the nearest 0.1 g (the same as for the lower resolution instrument), and so the total mass is written as 142.7 g.

Using judgement when multiplying values

Judgement is also needed in using the rule of thumb when multiplying values. Suppose you are calculating the masses of two blocks of aluminium, of volume 3.6 cm^3 and 4.2 cm^3. Multiplying by the density (2.7 g/cm^3) gives 9.72 g and 11.34 g respectively. All of the starting values have two significant figures, so applying the rule of thumb for the first block means that the calculated value is rounded to 9.7 g. This seems sensible.

However, applying the rule for the second block means rounding the value to 11 g, The only difference between the two calculations is that one gives a result a little under 10 g and the other a little over 10 g, though the first is rounded to the nearest 0.1 g and the second to the nearest 1 g. Here it may be more sensible to round the result for the second block to 11.3 g. Caution is needed to avoid over-rounding in such cases.

Thinking about the purposes of rounding

The above guidance applies to the *final result* of a calculation: in a multi-stage calculation, it is useful to retain *an extra significant figure* for the *intermediate values* that are calculated, in order to avoid rounding errors accumulating. On the other hand, rounding values to just *one significant figure* can be helpful if they are being used in a calculation to give an order of magnitude estimate.

Summary

There are no hard-and-fast rules for deciding on an appropriate number of significant figures. One difficulty is that this is linked to measurement uncertainty – a complex and subtle idea. However, that does not mean that 'anything goes', and the above discussion indicates some of the considerations for making sensible choices.

It is important that pupils should be able to identify the number of significant figures in a value, and to know how to round to a given number of significant figures. This is a matter of being correct or incorrect. Assessing how well they can round to appropriate numbers of significant figures involves finding out their reasons for doing so.

2.4 Calculating means

The ***mean*** of a set of values is the *sum* of the values divided by the *number* of values. (Strictly speaking, this is called the ***arithmetic mean***, to distinguish it from other means such as the *geometric mean*.) The arithmetic mean is so widely used that, in science, it is usually referred to as just the 'mean'. A common situation in school science for finding a mean is when taking repeated measurements in an experiment. The use of the term 'average' as an alternative to 'mean' should be avoided, since 'average' can be ambiguous. (See Section 6.5 *How big is a typical value?* on page 55 for further details about means and averages.)

The same considerations about significant figures apply to the calculation of means. For example, using the 'rule of thumb' when calculating the mean of the three measured values 7.5 cm, 7.8 cm and 7.6 cm gives a result of 7.6 cm. The sum of these numbers divided by 3 is 7.633 333, and the final result is given to *two* significant figures, since the original values have *two* significant figures. Note that the value '3' is an integer, and is *exactly* 3, so it is not treated as having one significant figure.

For a larger number of values, it may be justified for a mean to have a greater number of significant figures than the values of the data. For example, suppose you have 10 grapes and a balance reading to the nearest 1 g. The best way of finding the mean would be to put them all on the balance to get a total mass and divide by 10. But suppose instead that the mass of each grape is measured individually: 6 g, 5 g, 6 g, 7 g, 5 g, 5 g, 6 g, 6 g, 5 g and 6 g. The total is 57 g, and the mean would be 5.7 g.

Here it may be better *not* to round to 6 g, but to leave it as 5.7 g. The full explanation for this involves thinking about the possible range for the true value. Since the balance reads to the nearest 1 g, each measured value could be higher or lower than the true value by up to 0.5 g (e.g. a reading of 5 g means the true value is closer to this than to 4 g or 6 g, and lies between 4.5 g and 5.5 g). It is possible, though unlikely, that *all* the random measurement effects were working in the *same direction*. If all 10 measurements were too high, their total could be *higher* than the true value of the total by a maximum of 5 g (i.e. 10 × 0.5 g); if they were all too low, their total could be *lower* than the true value by a maximum of 5 g. Thus the true value of the total lies between 52 g and 62 g. These extremes, however, are very unlikely. It is much more probable that there will be some cancelling out of these random effects, with 57 g being the *best guess* of the total mass.

Another example of finding a mean is given in Section 6.2 *Variability and measurement uncertainty* on page 51, where the calculated value is rounded to *fewer* significant figures than the measured values, because there is a good deal of variability in the measurements.

These examples illustrate the difficulty in 11–16 science of providing hard-and-fast rules or full justifications for how to round to appropriate numbers of significant figures; it is best left to judgements about what seems to make good sense.

2.5 Index notation and powers

Pupils are most likely to come across ***index notation*** for the first time in the context of expressing the ***square*** of a number, for example that 3 × 3 can be expressed as 3^2 (and spoken as '3 squared'). In this example, the number '2' is called the ***index*** (or ***power*** or ***exponent***), and, in speech, the expression can also be read as '3 to the *power* of 2'. This can be extended to the ***cube*** of a number (e.g. 3^3, '3 cubed' or '3 to the power of 3') and to higher indices (e.g. 3^4, 3^5, 3^6, etc.).

The use of indices also applies to *units*. For example, the area of a piece of paper of size 20 cm by 10 cm can be expressed in units of cm^2.

$$\text{area of paper} = 20\,\text{cm} \times 10\,\text{cm} = 20 \times 10 \times \text{cm} \times \text{cm} = 200\,\text{cm}^2$$

Note that the unit is better pronounced 'square centimetres' rather than 'centimetres squared'. Saying '200 square centimetres' is unambiguous and gives a more direct sense of the area: saying '200 centimetres squared' could be interpreted as either 200 cm^2 or $(200\,cm)^2$, i.e. as 40 000 cm^2. Other common units used in science involving indices are m^2 ('square metres'), cm^3 ('cubic centimetres'), dm^3 ('cubic decimetres') and m^3 ('cubic metres').

The symbol √ is used for the ***square root*** of a number. For example, the square root of 9 can be written as $\sqrt{9}$. This has two values, 3 and –3 (also written as ±3), since both 3^2 and $(–3)^2$ are equal to 9. Similarly, the symbol for a ***cube root*** is $\sqrt[3]{}$, so $\sqrt[3]{27} = 3$. (Note that 27 has only one cube root, since $(–3)^3$ is –27 and not 27).

Roots may also be expressed using *fractional indices*, so the square root of 9 would be written as $9^{½}$, the cube root of 27 would be written as $27^{⅓}$, and so on. In science, the use of fractional indices may be encountered post-16, though it is not common at secondary level. The notion of a fractional index might seem odd at first: while 3^2 can be explained as meaning 3 × 3, what could $3^{½}$ mean? One way of thinking about this is to consider what happens to indices in multiplication. For example, $3^2 \times 3^3 = 3^5$ (3 × 3 × 3 × 3 × 3); the two indices are added together. In a similar way, $3^{½} \times 3^{½} = 3^1$ (i.e. 3). So, $3^{½}$ is the number which when multiplied by itself gives 3; in other words, it is the square root of 3.

The ***reciprocal*** of a number can also be represented using index notation. For example, the reciprocal of 2 is ½ , which can also be written as 2^{-1}. Similarly, the reciprocal of 2^2 (i.e. the reciprocal of 4, which is ¼) can be represented as 2^{-2}. Again, negative indices are not commonly used in secondary school science, except for powers of 10 as described below. Post-16 students would be expected to be familiar with the scientific convention of using negative indices in units, such as for velocity ($m\,s^{-1}$) or density ($g\,cm^{-3}$), but for younger pupils in science it is clearer if these are expressed as m/s or g/cm^3.

Like fractional indices, negative indices can also seem strange. Here, thinking about what happens during division can help. For example, 100 000 ÷ 1000 = 100 can be written as $10^5 \div 10^3 = 10^2$. Here the second index is subtracted from the first. Similarly, $1000 \div 100\,000 = {}^1/_{100}$ can be written as $10^3 \div 10^5 = 10^{-2}$.

2.6 Dealing with very large and very small values

The standard unit of length is the metre but lengths that are much larger or much smaller may be better expressed in different units using ***unit prefixes***. For example:

- the thickness of a coin (0.0015 m) is more clearly expressed in millimetres (1.5 mm)
- the distance between two towns (135 000 m) is more clearly expressed in kilometres (135 km).

This avoids having too many zeros, either before or after the decimal point. Large numbers can be written by leaving a space (not a comma) between every three digits, which makes them easier to read, though still not as clear as changing units.

In the SI system, there are unit prefixes covering a wide range of sizes, creating a 'ladder' with each step differing from the next by a factor of 1000 (or 10^3). Figure 2 shows the most commonly used unit prefixes.

Figure 2.1 Prefixes for SI units

Unit prefix	Unit prefix symbol	Multiplying factor			Example	
					Unit name	Unit symbol
tera-	T	1 000 000 000 000	or	10^{12}	terawatt	TW
giga-	G	1 000 000 000	or	10^9	gigawatt	GW
mega-	M	1 000 000	or	10^6	megawatt	MW
kilo-	k	1 000	or	10^3	kilowatt	kW
–	–	1	or	10^0	watt	W
milli-	m	0.001	or	10^{-3}	milliwatt	mW
micro-	μ	0.000 001	or	10^{-6}	microwatt	μW
nano-	n	0.000 000 001	or	10^{-9}	nanowatt	nW

In addition, two other prefixes are centi- (a hundredth) and deci- (a tenth), though these are only likely to be met as the centimetre (1 cm = 0.01 m) and the cubic decimetre ($1\,dm^3 = 0.001\,m^3 = 1000\,cm^3$).

Changing units can also help in comparing the sizes of values. For example, it is not easy to compare the masses of two objects expressed as 417 g and 1.24 kg. If they are both expressed in the same units, as 417 g and 1240 g, it may be easier to see that the second mass is about three times the first.

It is a common misconception that 'longer' numbers are bigger – the rule works for integers, but pupils may apply this inappropriately to any number. For example, when given the masses of two objects as 0.317 g and 0.52 g, pupils may think the first value is bigger ('317' is bigger than '52'). Converting the values to 317 mg and 520 mg makes the relative size clearer.

Another way of expressing very large or small values is to use ***standard form*** (also referred to as ***standard index form*** or ***scientific notation***). For example:

$127\,000$ *in standard form becomes* 1.27×10^5

In standard form, the first number has just one digit to the left of the decimal point (i.e. it is greater than or equal to 1 and less than 10); this is multiplied by a ***power of 10***.

One advantage of standard form is that it can make it easier to compare the ***orders of magnitude*** of very large or very small values. For example, 5.18×10^8 seconds is about 100 times bigger than 5.91×10^6 seconds. Written in full, the eye would be distracted by all the zeros; unless they were arranged one above the other, it would be hard to make the comparison. However, in order to make comparisons using standard form, pupils do need to be confident in using the notation. If they are not then comparison may be easier with the the numbers written in full.

Another advantage of using standard form is that it always makes clear the number of significant figures. An example given earlier was the problem of knowing the number of significant figures in the value 6300 m, and how this can be made clear by changing the units. Expressing in standard form is another way of showing this; for example, as 6.3×10^3 m (two significant figures) or 6.300×10^3 m (four significant figures).

It can also be easier to do calculations using standard form; for example, in multiplying 3.7×10^4 by 1.81×10^7. A calculator can be used to multiply 3.7 by 1.81 to give 6.697. Multiplying the powers of 10 can then be done mentally ($10^4 \times 10^7 = 10^{11}$) to give a final answer of 6.697×10^{11}. Using the numbers written in full on a calculator could easily lead to mistakes being made.

Multiplying 3.7×10^4 by 7×10^7 in the same way gives 25.9×10^{11} but this result is not in standard form, since 25.9 is greater than 10. When expressed in standard form, the result is 2.59×10^{12}.

Adding and subtracting numbers in standard form is trickier. The easiest way is to express them as ‘ordinary numbers’ and then carry out the calculation. The result can then be changed back to standard form.

Note that, when writing large numbers, it is now generally preferred in science to use a space rather than a comma as a ‘thousand separator’, i.e. to write 50 000 rather than 50,000. No separator is needed for numbers less than 10 000, i.e. 5000 rather than 5 000. The comma, however, is still the norm for everyday use in the UK. In many other countries, the comma has a different meaning: it is used as the ‘decimal mark’ instead of the dot used in the UK (e.g. 13,63 instead of 13.63).

2.7 Approximations and orders of magnitude

It is a useful habit when doing calculations to ask ‘Does this make sense?’ There are two things to consider – one is about the process of calculation and the other is about the ‘real-world’ values produced.

In both science and mathematics, pupils should be encouraged to use ***approximations*** so that they can check, for example, that when they use a calculator the output is roughly what they expect. They can do this by rounding all of the numbers in a calculation to one significant figure. For example, if the calculation is to multiply 36.9 by 6.2 then this becomes $40 \times 6 = 240$. The actual result is 228.78, which is close to the ***estimate***, but if they get 22 878 then they know that something has gone wrong. This number is the wrong ***order of magnitude***.

It is also important to think about whether a calculated value makes sense as an order of magnitude related to the real world; for example, a leaf of mass 3.97 kg, a temperature rise of water of 250 °C, or a car travelling down a motorway at 90 metres per hour. The first two values are far too large and the third is far too small. Being able to make such judgements requires pupils to have a sense of the magnitude of a range of units. Such an understanding can start early, for example with units of mass and length related to familiar objects, extending later to a wider range of values and to other quantities such as energy and power.